Non-comm defs & 3-fold flops

arXiv: 1309.0698 w/ Michael Wemyss (pron. Weemz) in Edinburgh

PLAN

1 Intro: birat geom

2 New invts: contractⁿ alg A_{con}

3 Results: A_{con} & NC deformatⁿs

4 Calculating new invts

5 Homological alg & twist functors

6 The future

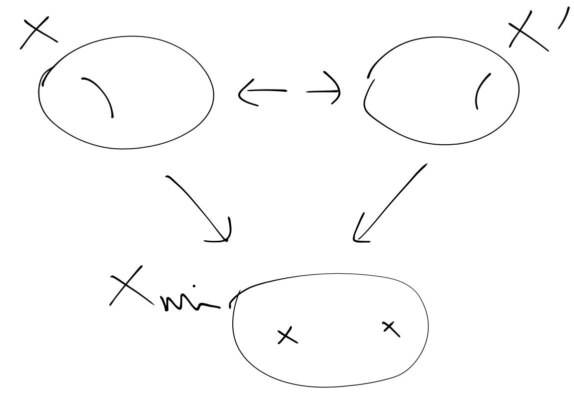
§1 Intro: birat geom

Q: For a smooth d-fold $X/\mathbb{C}$, which $X' \overset{\text{birat}}{\longleftrightarrow} X$?

A: gets harder as d increases!

$\boxed{d=1}$ $X' \simeq X$ ✓

$\boxed{d=2}$ $\exists$ blow-ups & contract^ns, eg.



Def^n X_{min} is <u>min. model</u> if $K.C \geq 0$ $\forall$ curves C

<u>Thm</u> [Castelnuovo] If X has min. model,

(1) it is unique,

(2) can obtain by contract^ns (of (-1)-curves).

$\boxed{d=3}$ smooth 3-fold

$$\begin{tikzcd} X \arrow[d, dashed] & X' \arrow[d, dashed] \\ X_{min} \arrow[r, dashed, leftrightarrow, "\text{flop}"'] & X'_{min} \end{tikzcd}$$

"run min. model programme" (contract^ns, flips)

min. model,

$\boxed{\text{iso in codim 2}}$

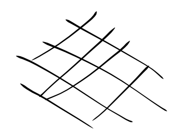
Zoology of 3-fold flops

eg 1 [Atiyah]

$\mathbb{P}^1 \cong C \subset X$ sm 3-fold, $\mathcal{N}_{C|X} \cong \mathcal{O}(-1)^{\oplus 2}$

$$\begin{tikzcd} & \mathrm{Bl}_C X \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, leftrightarrow, "\text{flop}"'] & & X' \end{tikzcd}$$

Exc. div. $= \mathrm{Tot}\,\mathbb{P}(\mathcal{N}_{C|X}) \cong \mathbb{P}^1 \times \mathbb{P}^1$



blow down other way

Rem: C rigid

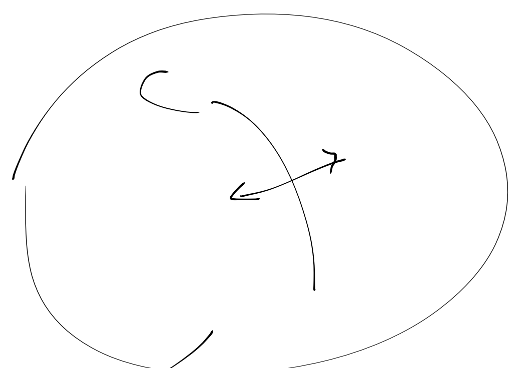
eg 2 As above, $\mathcal{N}_{C|X} \cong \mathcal{O}(-2) \oplus \mathcal{O}$

Learnt from [Reid ('83)]

min. models of can 3-folds

Pagoda paper

eg



C rigid, but not scheme-theoretically

infinitesimal deform: $\boxed{(n-1)\text{th order}}$

$$X = \mathrm{Bl}\left(\frac{\mathbb{C}[u,v,x,y]}{uv = x^2 - y^{2n}} \right) \qquad \boxed{n > 1}$$

eg 2 cont

$\boxed{\text{Def}^{\underline{n}}}$ width$(C) :=$ order of deform$^{\underline{n}}$ $+ 1$

Rem: width $= 1 \implies$ Atiyah

$$\begin{tikzcd}
 & \mathrm{Bl}_{\hat C}\hat X \arrow[dl] \arrow[dr] & \\
(-1,-1)\text{-curve } \hat C \subset \hat X \arrow[d, dotted] & & \hat X' \arrow[d, dotted] \\
(-2,0)\text{-curve } C \subset X \arrow[rr, dashed, leftrightarrow, "\text{flop}"'] & & X'
\end{tikzcd}$$

Atiyah

height n

$n-1$ blow-ups of sm. curves

width decreases

Notice (1) order of deform$^{\underline{n}}$ = heights of pagoda eg 3 [Laufer]

$$X = Bl_I\left(\frac{\mathbb{C}[u,v,x,y]}{u^2+v^2y = x(x^2+y^3)}\right) \supset \mathrm{Exc} =: C$$

Elementary (but not easy) to show

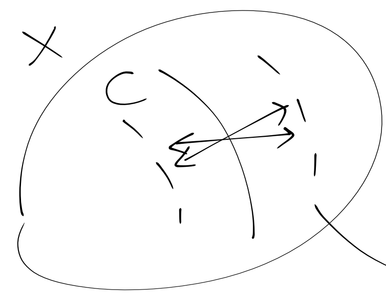
① $\mathcal{N}_{C|X} \simeq \mathcal{O}(-3)\oplus\mathcal{O}(1)$

② C floppable.

Infinitesimal defs of C in X controlled by

③ $\mathrm{Ext}^1(\mathcal{O}_C,\mathcal{O}_C) \simeq \mathbb{C}^2$ (from ①)

So looks something like:



curve moves in a plane (infinitesimally)

We'll improve on this

§2 New mts

Motivating question:

- how to generalize to $d>3$? $d \gg 3$?!

Deformat= f=tors

AIM: Consider formal NC defs of $E \in \mathcal{C}$, ab. cat

Algebras

Def= $\underline{1\text{-pointed } \mathbb{C}\text{-alg}}$ R is $\mathbb{C}$-alg R

w/ morphisms $\mathbb{C}^1 \xrightarrow{i} R \xrightarrow{p} \mathbb{C}^1$.

$$\mathbb{1} = p \circ i$$

Form category Alg_1

sim. $\mathrm{CommAlg}_1$ ⟵ $\boxed{\text{in here, } p \text{ gives } \mathbb{C}\text{-pt in } \operatorname{Spec} R}$

Def= $\mathrm{Art}_1 \subset \mathrm{Alg}_1$

$$\mathrm{Art}_1 = \{ R \mid \dim_{\mathbb{C}} R < \infty, \ \mathfrak{m} := \ker(p) \text{ nilpotent} \}$$

Rem $\mathrm{CommArt}_1$ = "fat pts"

Art_1 = "NC-fat points"

$\operatorname{Spec} R$

max ideal $\mathfrak{m}$

Functors

Reminder Usual deformations for $E \in QCoh(X)$

$$\mathrm{CommDef}_E^{QCoh(X)} : \mathrm{CommArt}_1 \to \mathrm{Sets}$$

$$(R, m) \longmapsto \left\{ \mathcal{F}, \delta \;\middle|\; \begin{array}{l} \mathcal{F} \text{ flat } \mathrm{Spec}(R) \text{ family in } QCoh(X) \\ \mathcal{F}|_m \xrightarrow[\delta]{\sim} E \end{array} \right\}$$

i.e. $\mathcal{F} \in QCoh(X \times \mathrm{Spec}(R))$

cf. $QCoh(X)^R \leftarrow$ sheaf on X w/ R-module structure.

This generalizes to give:

$\boxed{\text{Defn}}$ $\mathrm{Def}_E^{\mathcal{C}} : \mathrm{Art}_1 \to \mathrm{Sets}$ deformation functor for $E \in \mathcal{C}$

[cf. Lowen, Eriksen, E. Segal, ELO ...]

$$(R, m) \longmapsto \left\{ \underset{R\text{-mods in } \mathcal{C}}{\mathcal{F} \in \mathcal{C}^R}, \delta \;\middle|\; \begin{array}{l} \mathcal{F} \otimes_R \_ : R\text{-mod} \to \mathcal{C} \text{ exact} \\ \mathcal{F} \otimes_R \frac{R}{m} \xrightarrow[\delta]{\sim} E \end{array} \right\}$$

Rem Can no longer take Spec in Art^1

simpler than CommDef!

Ex restriction $\mathrm{CommDef}_E^{\mathcal{C}} : \mathrm{CommArt}_1 \to \mathrm{Sets}$

$\cong$ usual deformations.

Contraction algebra A_{con}

Setup 3-fold (normal, qproj)

$$\begin{tikzcd} X \arrow[r, "\text{contraction}", "f"'] & X_{con} = \operatorname{Spec} R \\ C \arrow[u, phantom, "\cup" description] \arrow[r] & pt \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

proj, birat, $f_* \mathcal{O} \simeq \mathcal{O}$

s.t. $C^{red} \simeq \mathbb{P}^1$

e.g. flopping contraction ~~flipping~~

[vdB] $\exists$ tilting gen $\mathcal{B}$ on X (rel X_{con})

$\mathcal{B} = \mathcal{O} \oplus M$

M — universal extension if not type A.

$\rightsquigarrow$ $D^b(X) \simeq D^b(A\text{-mod})$

$A := \operatorname{End}_R(\mathcal{O} \oplus M)$

Can describe by quiver:

$$\begin{tikzcd} e_0 \arrow[loop left] \arrow[r, bend left] & e_1 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

— idempotents (projections to summands)

Def^n $A_{con} := A / [e_0]$ — two-sided ideal

$$\begin{tikzcd} e_1 \arrow[loop right] \end{tikzcd}$$

Example Show Table 2

eg 1

$$\mathcal{B} = \mathcal{O} \oplus \mathcal{O}(1)$$

x, y — $\mathbb{P}^1_{x:y}$

— fibre co-ords

eg 3 quantum cusp

most examples not type A $\rightsquigarrow$ non-comm

§3 RESULTS Link with deformation theory

Setup 3-fold (normal, q proj)

$$\begin{tikzcd} X \arrow[r, "\text{contraction}", "f"'] & X_{con} \\ \cup & \cup \\ C \arrow[r] & pt \end{tikzcd}$$

proj, birat, $f_* \mathcal{O} \simeq \mathcal{O}$

s.t. $C^{red} \simeq \mathbb{P}^1$

e.g. ~~flopping~~ contraction
flipping ''

2.12(1) 5.7

$\boxed{\text{Thm}}$ ① $A_{con} \in AA$, well-defined (fin. dim., self-inj)

② A_{con} represents $\mathrm{Def}^{QCoh\,X}_{E := \mathcal{O}_C(-1)}$ ⟵ cf vertex simple

$\boxed{\text{Cor}}$ $\exists$ universal family $\mathcal{E}$ / base A_{con} deforming E in X.
i.e. $\mathcal{E} \in (QCoh\,X)^{A_{con}}$

3.9

$\boxed{\text{Prop}}$ $A_{con} \simeq \mathrm{End}_X(\mathcal{E})$ forgets A_{con}-module struct

Rel$^{\underline{n}}$ with comm geom

$\boxed{\text{Def}^{\underline{n}}}$ $A^{ab}_{con} := A_{con} /$ commutators

$\underline{\text{Cor}}$? $\exists$ universal family $\mathcal{E}^{ab}$ / base $\mathrm{Spec}(A^{ab}_{con})$ defining E in X.

(i.e. reps $\mathrm{CommDef}^{2\mathrm{Coh}\,X}_{E}$)

$\underline{\text{Prop}}$ $A^{ab}_{con} \cong \mathrm{End}_X(\mathcal{E}^{ab})$

$\underline{\text{Rem}}$ consider A_{con} as an NC-scheme [Kapranov]

$$\begin{array}{c} \text{NC-scheme} \\ \big\downarrow \text{abelianize} \\ \mathrm{Spec}(A^{ab}_{con}) \end{array}$$

§4 Calculating new invts

Def^n (nc Width) $\mathrm{wid}(C) := \dim_{\mathbb{C}} A_{con}$

$\mathrm{wid}^{ab}(C)$ sim.

eg Type A, $\mathrm{wid}(C) = \mathrm{wid}^{ab}(C) =$ Reid's width.
Otherwise, may differ or not defined.

eg Laufer $A_{con} = \dfrac{\mathbb{C}\langle x,y\rangle}{\{x,y\},\ x^2=y^3}$ quantum cusp

anti-comm

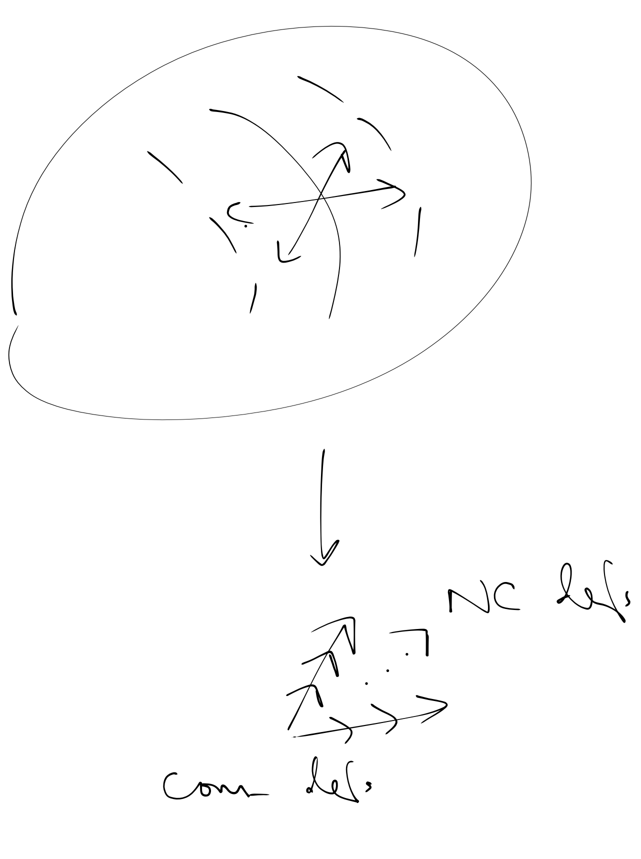
$$\{x,y^3\}=0 \qquad x^3 = xy^3 = y^3x$$

$$\Rightarrow x^3 = 0$$

In fact, have basis

$$\begin{matrix} 1 & \cdots & x^2 \\ \vdots & & \vdots \\ y^2 & \cdots & x^2y^2 \end{matrix} \Rightarrow \mathrm{wid}(C) = 9$$

$$A^{ab}_{con} = \frac{\mathbb{C}[x,y]}{xy,\ x^2=y^3} \Rightarrow \mathrm{wid}^{ab}(C) = 5$$



Rem $\exists$ discrete fam. w/ $y^3 \longrightarrow y^{2n+1}$

- All type D_4
- $\mathrm{wid}(C) = 3(2n+1)$

Show Table 3.

§5 Homological algebra & twist functors

Suppose
$$\begin{tikzcd} X \arrow[rr, dashed, leftrightarrow, "\text{flop}"] \arrow[dr] & & X' \arrow[dl] \\ & X_{con} & \end{tikzcd}$$
both proj, sings $\leq$ term. Gor.

Thm [Bridgeland, Chen] $\quad F : D^b(X) \xrightarrow{\sim} D^b(X')$

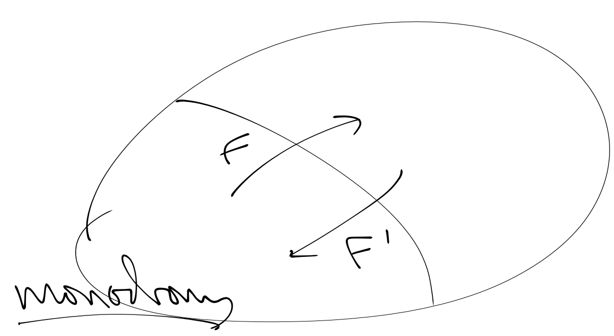
sing. case

Fourier–Mukai corresp $X \times_{con} X'$

Using symmetry, have flop-flop functor $FF : D^b(X) \circlearrowleft$, $\quad FF = F' \circ F$

Turns out to be $\boxed{\text{non-trivial}}$

Stability m'fld heuristic



Result D-monodromy FF controlled by A_{con} (<u>not</u> by A^{ab}_{con})

Thm ③ $FF \simeq$ sph twist $T_{A_{con}}$ about $\mathcal{E}/A_{con}$
[..] $\simeq \left\{ \mathcal{E} \otimes_{A_{con}} \underbrace{\mathrm{Hom}_X(\mathcal{E}, -)}_{\text{module struct}} \longrightarrow (-) \right\}$

Rem Generalizes work of Toda in type A.

Prop twist $T^{ab}_{A_{con}}$ not an autoequiv

Sph. functors see [Seidel-Thomas, Khov.-Thomas, Anno-Logvinenko ...]

Propose $\mathcal{E} \overset{\mathbb{L}}{\otimes}_{A_{con}} \_ : D(A_{con}\text{-mod}) \longrightarrow D(X)$ sph

Warning: non-comm
non-finite gl. dim
$\exists$ duality theory ...

§6 The future

Q: What structure ~~does~~ should moduli of DCoh(CY3) have?

A: derived, higher stack, (−1)-shifted symplectic

See [Lurie, Toën, PTVV, Brav-Bussi-Joyce...]

A: NC-scheme, and indeed ...

cf [Kapranov, Efimov-Lunts-Orlov]

"Multi-points"

Form category Alg_n (cf. Alg_1 earlier)

subcat Art_n

Rem algebras have n orthogonal idempotents.

Picture family of n NC-fat points

interactions!

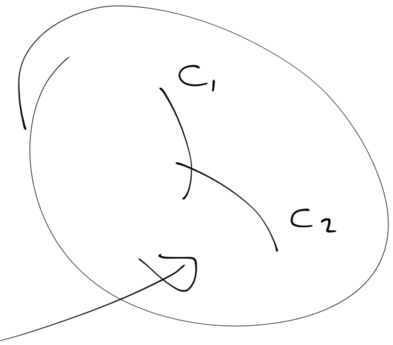
trel$\hat{f}$s so fin. dim.

Two-pointed example:

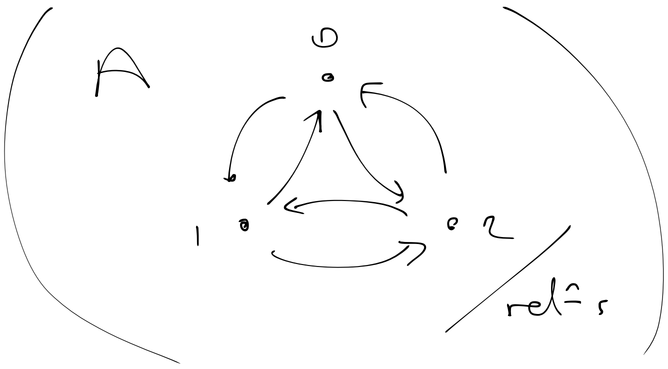
$$uv = \underbrace{f_1 f_2 f_3}_{\text{linear in } x, y}$$

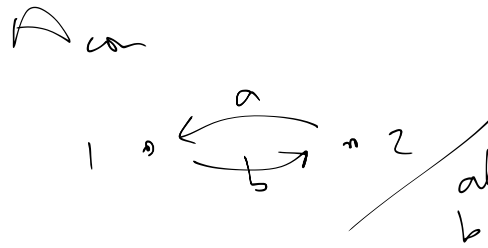
$f_i = x + \lambda_i y$
(all diff)

resol^n:



flop
together
or separately





$aba = 0$
$bab = 0$

Have $A_{con} \in \mathrm{Art}_2$

$$E = E_1 \oplus E_2 = \mathcal{O}_{C_1}(-1) \oplus \mathcal{O}_{C_2}(-1) \in D^b(X)^{\mathbb{C}^2}$$

$$\mathcal{E}_{12} \in \langle E_1, E_2 \rangle \subset D^b(X)^{A_{con}}$$

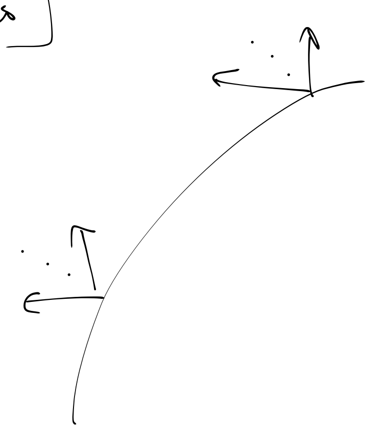
(think $\mathbb{C}^2 \in \mathrm{Art}_2$)

<u>Results</u> $FF_{\text{both!}} = T_2$

<u>Conclude</u> moduli has NC $\mathbb{C}$-pts
multi-pointed
generic points?

We have <u>not</u> <u>discussed</u>:

1. Many more examples, most yet to be calculated
   - computer alg.
2. Flips
   - Francia (comm, defs =) (3.13)
3. A_{con} self-injective, f. CY0
   - algebraic interest
4. When does curve C flop? flip?
5. $\dim X > 3$
   - fam. twist [Horja, Anno-Logvinenko]
   - what NC geom occurs?
6. general Q:
   - given object $E \in \mathbb{D}$ (der. cat) when do its <u>deformations</u> yield <u>symms</u>?



<u>Final Obs</u> Try to understand $D^b(X)$, classical X <u>forced</u> to use <u>non-comm</u> geom.

THANKS!